SELF-SIMILAR SOLUTIONS OF DIFFUSION PROBLEMS OF A MAGNETIC FIELD IN A CONDUCTOR

A. V. Chervyakov

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The diffusion problem of a magentic field pulse in a conductor was formulated and investigated by many authors [1-3]. As to the electric conductivity of the medium, one usually assumes satisfaction of one of the three hypotheses: 1) $\sigma = \sigma_0 = \text{const}$, 2) $\sigma = 1/AQ^\beta$, $\beta = \text{const}$, $A = \text{const}$ (Q is the increase in internal energy with respect to the initial state), 3) $\sigma = \sigma_0/(1 + \beta Q)$. The thermal conductivity of the medium was taken to be constant in these studies. Hypothesis 2 was treated in [2, 4], and a problem with σ dependence of shape 3 – in [3]. A constant conductivity was investigated in [1, 5]. The complexity of the problem in the case of hypotheses 2, 3 is dictated by the nonlinear dependence of conduction of the conductor on its thermodynamic characteristics. Even in the case of constant σ, however, investigators have resorted to further simplifications of the model. For example, the thermal conductivity of the medium is not taken into account in [1], and in [5] the boundary regime is treated only in the form of a magnetic field discontinuity. An analytic solution of the linear problem is given in the present paper with account of the thermal conductivity and with a graduated boundary regime of the magnetic field, and the nonlinear problem is investigated for hypothesis 2, as well as the structure of the solution for hypothesis 3.

1. As is well known [1, 2], the penetration of a magnetic field into an incompressible conductor of planar or cylindrical geometry is described by the system of equations

$$\frac{\partial H}{\partial x} = -j, \quad \frac{\partial E}{\partial x} + \mu\frac{\partial H}{\partial t} = -\frac{n_0 E}{x}, \quad j = \sigma(Q)E, \quad \frac{\partial Q}{\partial t} = \frac{j^2}{\sigma} - \frac{\partial q}{\partial x}, \quad q = -\varkappa\frac{\partial Q}{\partial x}. \tag{1.1}$$

For $n_0 = 1$ we obtain the diffusion problem in a cylindrical conductor, and for $n_0 = 0$ we have the one-dimensional planar problem. In the equations above $\varkappa$ is the thermal conductivity coefficient, assumed constant in the present model, and the remaining notations are the common ones.

The magnetic field $H(x_0, t) = H_0 t^\alpha$ $(\alpha \geqslant 0)$ and the thermal flux $q(x_0, t) = 0$ are given on a planar or cylindrical boundary of the conductor, and the diffusion process of the magnetic field is investigated for vanishing initial values of H, E, Q, q.

2. Consider the problem with $\sigma = \sigma_0$. For $x_0 = 0$ the problem conditions make it possible to introduce the self-similar variable $z = x/t^{1/2}$. In dimensionless variables we obtain the following boundary value problem for the system of ordinary differential equations:

$$\begin{gathered}\frac{dH}{dz} = -E, \quad \frac{dE}{dz} = -\alpha H - \frac{z}{2}E - \frac{n_0 E}{z},\\ \frac{dq}{dz} = -2\alpha Q - \frac{zq}{2\varkappa} + E^2, \quad \varkappa\frac{dQ}{dz} = -q,\\ H(0) = 1,\ q(0) = 0,\ H(\infty) = Q(\infty) = 0.\end{gathered} \tag{2.1}$$

The first two equations of system (2.1) are solved independently of the remaining ones. Eliminating E and putting $\zeta = z^2/4$, we have for H

$$\zeta H_{\zeta\zeta} + \left(\zeta + \frac{n_0+1}{2}\right)H_\zeta - \alpha H = 0. \tag{2.2}$$

Putting $H(\zeta) = \int\limits_C e^{p\zeta}V(p)\,dp$, from (2.2) we find that $V(p) \sim (p+1)^{\alpha+(n_0-1)/2}\, p^{-1-\alpha}$. The two independent solutions of Eq. (2.2) are obtained in the form

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$$H_1 \sim \int_{C_1} e^{\zeta p}(1+p)^{\alpha+(n_0-1)/2} p^{-1-\alpha}\, dp,$$

$$H_2 \sim \int_{C_2} e^{\zeta p}(1+p)^{\alpha+(n_0-1)/2} p^{-1-\alpha}\, dp.$$

The contour C_1 starts from $-\infty$, bypasses the point -1 in the positive direction, and is removed to $-\infty$. The contour C_2 is constructed similarly, except that along with the point -1 it also bypasses the point 0. Taking into account the condition $H(z_0) = 1$ [in system (2.1) $z_0 = 0$], as well as the asymptotic $H_2(\zeta)$ for $\zeta \to \infty$ [$\beta = (n_0 - 1)/2$]

$$H_2(\zeta) \sim \frac{\zeta^\alpha}{\Gamma(1+\alpha)} + \sum_{k=1}^{\infty} \frac{(\alpha+\beta-1)\dots(\alpha+\beta-k)}{k!} \frac{\zeta^{\alpha-k}}{\Gamma(1+\alpha-k)},$$

the electromagnetic field is found in the form

$$\begin{aligned} H(z) &= \int_1^\infty e^{-\frac{z^2 y}{4}} (y-1)^{\alpha+(n_0-1)/2} y^{-1-\alpha}\, dy \Bigg/ \int_1^\infty e^{-\frac{z_0^2 y}{4}} (y-1)^{\alpha+(n_0-1)/2} y^{-1-\alpha} dy, \\ E(z) &= \frac{z}{2} \int_1^\infty e^{-\frac{z^2 y}{4}} (y-1)^{\alpha+(n_0-1)/2} y^{-\alpha} dy \Bigg/ \int_1^\infty e^{-\frac{z_0^2 y}{4}} (y-1)^{\alpha+(n_0-1)/2} y^{-1-\alpha}\, dy. \end{aligned} \tag{2.3}$$

It follows from Eqs. (2.3) that $E(0) = \Gamma(1 + \alpha)/\Gamma(1/2 + \alpha)$. Transforming to the original variables of system (1.1), we obtain the current density j at the conductor boundary (x = 0):

$$j(0, t) = \sigma_0 t^{\alpha-1/2} \frac{\Gamma(1+\alpha)}{\Gamma\left(\frac{1}{2}+\alpha\right)}.$$

The last equation shows that the current density discontinuity vanishes for $\alpha \geqslant 1/2$. In the nonlinear problem (see hypotheses 2 and 3) the current density discontinuity at $t \to 0$ is controlled not only by the boundary regime, but also by the thermal conductivity coefficient. For hypothesis 3, $t \to 0$ $\sigma \to \sigma_0$, and the behavior of the current density is the same as in the linear problem [see (4.2)]. For $\sigma = 1/AQ^\beta$ the structure of the solution differs sharply from hypotheses 1, 2 (as shown below), but here the problem of the behavior of j(0, t) as a function of $\varkappa$ is not touched upon. The last two equations of system (2.1) reduce to

$$\varkappa Q'' + \frac{z}{2} Q' - 2\alpha Q = -E^2(z) = -f\left(\frac{z^2}{4}\right).$$

Following the replacement $\zeta = z^2/4\varkappa$ we find

$$\zeta Q_{\zeta\zeta} + (\zeta + 1/2) Q_\zeta - 2\alpha Q = -f(\varkappa\zeta). \tag{2.4}$$

The solution of the corresponding homogeneous equation is selected in the form

$$Q = C_1 Q_1 + C_2 Q_2,$$

where

$$Q_1(\zeta) = \frac{1}{2\pi i} \int_C e^{\zeta p} (1+p)^{2\alpha-1/2} p^{-1-2\alpha}\, dp;$$

$$Q_2(\zeta) = \int_1^\infty e^{-\zeta y} (y-1)^{2\alpha-1/2} y^{-1-2\alpha}\, dy.$$

The contour C is a neighborhood with center at the point p = 0 and a radius of unity. One easily obtains asymptotic equations for $Q_1(\zeta)$ and $Q_2(\zeta)$ for $\zeta \to \infty$;

$$\begin{aligned} Q_1(\zeta) &= \frac{\zeta^{2\alpha}}{\Gamma(1+2\alpha)} + \sum_{k=1}^{n} \frac{(2\alpha-1/2)\dots(2\alpha+1/2-k)}{k!} \frac{\zeta^{2\alpha-k}}{\Gamma(1+2\alpha-k)} + O\left(\zeta^{2\alpha-n-1}\right), \\ Q_2(\zeta) &= \frac{e^{-\zeta}}{\zeta^{2\alpha+1/2}} \left(\Gamma(2\alpha+1/2) + \sum_{n=1}^{p} (-1)^n \frac{n(1+2\alpha)\dots(n+2\alpha)}{n!\,\zeta^n} \Gamma(2\alpha+1/2+n) + O\left(\zeta^{-p-1}\right)\right). \end{aligned} \tag{2.5}$$

For $\zeta \to 0$ we have convergent series for Q_1 and Q_2

$$Q_1 \sim 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(n-1-2\alpha)\dots(1-2\alpha)(-2\alpha)}{n!\,(n-1/2)\dots(3/2)(1/2)} \zeta^n,$$

$$Q_2 \sim \sqrt{\zeta}\left(1+\sum_{n=1}^{\infty}(-1)^n \frac{(n-2\alpha-1/2)\dots(1-2\alpha-1/2)}{n!\,(n+1/2)\dots 3/2}\zeta^n\right)+$$

$$+\frac{\sqrt{\pi}}{2}2^{\frac{4\alpha-1}{2}}\int_0^{\pi}(1+\cos\varphi)^{\frac{4\alpha-1}{4}}\cos\left(\frac{4\alpha+1}{4}\varphi\right)d\varphi\left(1+\sum_{n=1}^{\infty}(-1)^n\frac{(n-1-2\alpha)\dots(1-2\alpha)(-2\alpha)}{n!\,(n-1/2)\dots(3/2)(1/2)}\zeta^n\right).$$

Consider initially planar geometry. In this case $n_0 = 0$ in Eqs. (2.3). The general solution of Eq. (2.4) can be written in the form

$$Q = C_1Q_1 + C_2Q_2 + Q_1\int_0^{\zeta}\frac{Q_2(\xi)f(\varkappa\xi)}{\xi W(\xi)}d\xi - Q_2\int_0^{\zeta}\frac{Q_1(\xi)f(\varkappa\xi)}{\xi W(\xi)}d\xi,\quad W(\xi) = Q_1Q_2' - Q_2Q_1'.$$

The constants C_1 and C_2 are determined from the boundary conditions $\sqrt{\zeta}Q_\zeta' \to 0$ $(\zeta \to 0)$, $Q(\infty) = 0$. Taking this into account, we finally obtain

$$Q = -Q_1(\zeta)\int_{\zeta}^{\infty}\frac{Q_2(\xi)f(\varkappa\xi)}{\xi W(\xi)}d\xi - Q_2(\zeta)\int_0^{\zeta}\frac{Q_1(\xi)f(\varkappa\xi)}{\xi W(\xi)}d\xi =$$

$$= -Q\left(\frac{z^2}{4\varkappa}\right)\int_z^{\infty}\frac{Q_2\left(\frac{y^2}{4\varkappa}\right)f\left(\frac{y^2}{4}\right)}{yW\left(\frac{y^2}{4\varkappa}\right)}dy - Q_2\left(\frac{z^2}{4\varkappa}\right)\int_0^{z}\frac{Q_1\left(\frac{y^2}{4\varkappa}\right)f\left(\frac{y^2}{4}\right)}{yW\left(\frac{y^2}{4\varkappa}\right)}dy. \qquad (2.6)$$

Equation (2.6) is convenient for further study. Putting $\varkappa \to 0$ and taking into account the asymptote (2.5), we find the expansion in $\varkappa$. In particular, for $\varkappa = 0$ (the zeroth approximation) we have

$$Q(z) = 2z^{4\alpha}\int_z^{\infty}\frac{f\left(\frac{y^2}{4}\right)}{y^{1+4\alpha}}dy \underset{z\to 0}{\to}\begin{cases}+\infty, & \alpha = 0,\\ \frac{1}{2\alpha}\frac{\Gamma^2(1+\alpha)}{\Gamma^2(\alpha+1/2)}, & \alpha > 0.\end{cases}$$

We now consider cylindrical geometry ($n_0 = 1$). To conserve the self-similarity of the problem we assume, as was done in [1, 5], that the cylindrical boundary of the conductor undergoes a phase transition, in which the surface of the conductor evaporates. In this case the boundary regime must be given in the form of a step function ($\alpha = 0$). The general solution of Eq. (2.3) is represented in the form

$$Q(\zeta) = C_1Q_1 + C_2Q_2 + Q_1(\zeta)\int_{\zeta_0}^{\zeta}\frac{Q_2(\xi)f(\varkappa\xi)}{\xi W(\xi)}d\xi - Q_2(\zeta)\int_{\zeta_0}^{\zeta}\frac{Q_1(\xi)f(\varkappa\xi)}{\xi W(\xi)}d\xi.$$

The following requirements are generated for $\zeta = \zeta_0$:

$$\left.\frac{dQ}{d\zeta}\right|_{\zeta=\zeta_0} = 0,\quad Q(\zeta_0) = Q^*$$

(Q^* is the amount of heat required to heat a unit volume of the conductor from the initial temperature to the boiling temperature and for its total evaporation). These conditions made it possible to determine $\zeta_0(z_0)$ from the relation

$$Q^* = -\frac{W(\zeta_0)}{Q_2'(\zeta_0)}\int_{\zeta_0}^{\infty}\frac{Q_2(\xi)f(\varkappa\xi)}{\xi W(\xi)}d\xi. \qquad (2.7)$$

Since $\alpha = 0$, expression (2.7) undergoes further simplification. Indeed, in this case $Q_1 = 1$, $W(\xi) = Q_2'(\xi)$, and, taking into account the asymptote (2.5), we find that for $\zeta \to \infty$ $Q_2(\zeta)/W(\zeta) \to -1$. If $\zeta_0 \to \infty$ in relation (2.7), there exists a limit to its right-hand side:

$$\lim_{\zeta_0\to\infty}\int_{\zeta_0}^{\infty}\frac{f(\varkappa\xi)}{\xi}d\xi = \frac{1}{2}.$$

To obtain this result we used Eqs. (2.3) and took into account that $f(z^2/4) = E^2(z, z_0)$. Consequently, the maximum magnetic field which can be supported by the cylindrical surface can be determined from the relation connecting the dimensional and dimensionless Q^*, i.e., $Q^*_{dim} = \mu H_0^2 Q^*$:

$$H_{0\max}=\sqrt{\frac{2Q^*}{\mu}}. \qquad (2.8)$$

3. We investigate the nonlinear problem with the dependence $\sigma = 1/AQ^{\beta}$, β = const > 0, neglecting the thermal conductivity. For $x_0 = 0$ the conditions of the problem make it possible to introduce the self-similar variable $z = xt^{-(1+2\alpha\beta)/2}$. We obtain a system of ordinary differential equations

$$\frac{dH}{dz}=-EQ^{-\beta},\quad \frac{dE}{dz}=-\alpha H+\frac{1+2\alpha\beta}{2}zEQ^{-\beta}-\frac{n_0E}{z},$$

$$\frac{dQ}{dz}=\frac{2z}{1+2\alpha\beta}\left(2\alpha Q-E^2Q^{-\beta}\right),\quad H(0)=1,$$

in which we replace variables according to the equations

$$\zeta=\frac{z_0-z}{z_0},\quad H=\sqrt{\frac{\gamma}{\beta}}\,2\beta z_0\left(2\beta\gamma z_0^2\right)^{(1-\beta)/2\beta}\zeta^{1/2\beta}h(\zeta),$$

$$E=\sqrt{\frac{\gamma}{\beta}}\left(2\beta\gamma z_0^2\right)^{(1+\beta)/2\beta}\zeta^{1/2\beta}e(\zeta),$$

$$Q=\left(2\beta\gamma z_0^2\right)^{1/\beta}\zeta^{1/\beta}q(\zeta),\quad \gamma=\frac{1+2\alpha\beta}{2}.$$

Following some calculations, we have equations for the functions h, e, q:

$$\zeta\frac{dh}{d\zeta}=\frac{1}{2\beta}\left(\frac{e}{q^{\beta}}-1\right),$$
$$\zeta\frac{de}{d\zeta}=\frac{1-\zeta}{2\beta}\frac{e}{q^{\beta}}+\left(\frac{n_0\zeta}{1-\zeta}-\frac{1}{2\beta}\right)e+\frac{\alpha}{\gamma}\zeta h, \qquad (3.1)$$
$$\zeta\frac{dq}{d\zeta}=\frac{1}{1-\zeta}\frac{1}{\beta}\frac{e^2}{q^{\beta}}-\frac{q}{\beta}-\frac{2\alpha}{\gamma}\frac{\zeta}{1-\zeta}q.$$

At $\zeta = 0$ we seek a holomorphic solution of system (3.1). The initial conditions are taken to be

$$h(0)=1,\ e(0)=1,\ q(0)=1.$$

Thus, we put $h = 1 + \tilde{h}$, $e = 1 + \tilde{e}$, $\tilde{q} + 1 = q$, where the functions $\tilde{h}$, $\tilde{e}$, $\tilde{q}$ are holomorphic at $\zeta = 0$ and acquire vanishing values. These functions are determined from the system of equations

$$\zeta\frac{d\tilde{h}}{d\zeta}=\frac{1+\tilde{e}}{2\beta}\sum_{n=1}^{\infty}(-1)^n\frac{\beta(\beta+1)\ldots(\beta+n-1)}{n!}\tilde{q}^n+\frac{\tilde{e}-\tilde{h}}{2\beta},$$

$$\zeta\frac{d\tilde{e}}{d\zeta}=\frac{(1-\zeta)(1+\tilde{e})}{2\beta}\sum_{n=1}^{\infty}(-1)^n\frac{\beta(\beta+1)\ldots(\beta+n-1)}{n!}\tilde{q}^n+\zeta(1+\tilde{e})\left(\frac{n_0}{1-\zeta}-\frac{1}{2\beta}\right)+\frac{\alpha}{\gamma}\zeta(1+\tilde{h}),$$

$$\zeta\frac{d\tilde{q}}{d\zeta}=\frac{(1+\tilde{e})^2}{\beta(1-\zeta)}\sum_{n=0}^{\infty}(-1)^n\frac{\beta(\beta+1)\ldots(\beta+n-1)}{n!}\tilde{q}^n+\frac{\tilde{e}(2+\tilde{e})+\zeta-\tilde{q}(1-\zeta)}{\beta(1-\zeta)}-\frac{2\alpha}{\gamma}\frac{\zeta}{1-\zeta}(1+\tilde{q}).$$

We substitute integral series for $\tilde{h}$, $\tilde{e}$, $\tilde{q}$, i.e., we put

$$\tilde{h}=\sum_{n=1}^{\infty}h_n\zeta^n,\quad \tilde{e}=\sum_{n=1}^{\infty}e_n\zeta^n,\quad \tilde{q}=\sum_{n=1}^{\infty}q_n\zeta^n.$$

The following system of linear equations is obtained for finding the coefficients

$$(-1)^n\beta q_n+e_n-(1+2n\beta)h_n=0,$$
$$(1+2n\beta)e_n-(1+2n\beta)h_n=\ldots,$$
$$(1+\beta n)q_n-e_n-(1+2n\beta)h_n=\ldots$$

(the dotted lines denote known quantities, or quantities determined at preceding steps). The determinant of this system is nonvanishing for $\beta > 0$. The convergence of the series obtained is determined by the method of upper bound functions. The problem arises of the radius of convergence of these series. If it is less than unity, we continue analytically the solution constructed along the real ζ axis to the point $\zeta = 1$. Analysis of system (3.1) shows that such a continuation is indeed possible. It follows from (3.1) that $\tilde{q}(\zeta) \to \infty$ for $\zeta \to 1$. Consider a system of two equations in z_0, ζ_0:

$$\sqrt{\frac{\gamma}{\beta}}\, 2\beta z_0 \left(2\beta\gamma z_0^2\right)^{(1-\beta)/2\beta} \zeta_0^{1/2\beta} h(\zeta_0) = 1,$$

$$\left(2\beta\gamma z_0^2\right)^{1/\beta} \zeta_0^{1/\beta} q(\zeta_0) = Q^*.$$

This system can always be solved for $n_0 = 0, 1$. As a result of solving it for $\alpha = 0$ we obtain two curves: $x_1 = z_0(1 - \zeta_0)\sqrt{t}$, $x_2 = z_0\sqrt{t}$. The first describes the evaporation front of the surface of the conductor, and the second – the propagation front of the boundary regime. As is well known, in the linear problem with $\varkappa = 0$ the boundary regime propagates with an infinite velocity. An infinite front velocity also occurs in the case of hypothesis 3. A dependence of the electric conductivity by the equation $\sigma = 1/AQ^\beta$ models a vanishing resistance for $t \to 0$, i.e., the presence of a superconducting phase at the initial moment of time. The analysis provided in Sec. 3 shows that in the given model one observes a finite propagation velocity of electromagnetic perturbations. For nonlinear parabolic problems such solutions with a finite propagation velocity of perturbations were first noted in [6].

4. Let $\sigma = \sigma_0/(1 + \beta Q)$. For $\alpha = 0$ one can introduce the self-similar variable $z = xt^{-1/2}$. The system of ordinary differential equations is in this case

$$\frac{dE}{dz} = -\frac{z}{2}\frac{E}{1+\beta Q} - \frac{n_0 E}{z}, \quad \frac{dH}{dz} = -\frac{E}{1+\beta Q},$$
$$\varkappa\frac{dq}{dz} = -\frac{z}{2}q + \frac{E^2}{1+\beta Q}, \quad \frac{dQ}{dz} = -q, \quad H(0) = 1, \quad q(0) = 0. \tag{4.1}$$

We represent the required solution of (4.1) in the form of integral series in the parameter β:

$$H = \sum_{n=0}^{\infty} H_n\beta^n, \quad E = \sum_{n=0}^{\infty} E_n\beta^n, \ldots$$

In the zeroth approximation we have

$$H_0' = -E_0, \quad E_0' = -\frac{z}{2}E_0 - \frac{n_0 E_0}{z},$$
$$\varkappa q_0' = -\frac{z}{2}q_0 + E_0^2, \quad Q_0' = -q_0, \quad H_0(0) = 1, \quad q_0(0) = 0. \tag{4.2}$$

These equations coincide with (2.1) for $\alpha = 0$. The subsequent approximations are determined from the inhomogeneous system of ordinary differential equations:

$$H_n' = -E_n - \sum_{p=1}^{n} Q_{p-1}H_{n-p}',$$
$$E_n' = -\frac{z}{2}E_n - \frac{n_0}{z}E_n - \sum_{p=1}^{n} Q_{p-1}\left(E_{n-p}' + \frac{n_0}{z}E_{n-p}\right),$$
$$\varkappa q_n' = -\frac{z}{2}q_n - \sum_{p=1}^{n} Q_{p-1}\left(\varkappa q_{n-p}' + \frac{z}{2}q_{n-p}\right) + \sum_{p=0}^{n} E_pE_{n-p},$$
$$Q_n' = -q_n, \quad H_n(0) = 0, \quad q_n(0) = 0, \quad n = 1, 2, 3, \ldots \tag{4.3}$$

For $n_0 = 0$ the problem (4.3) can be solved for each n. The convergence of the series obtained is established by the method of upper bound functions. For $n_0 = 1$ it is necessary to treat the phase transition problem, as was done in Secs. 2, 3.

It is of large practical value to find the values of the maximum magnetic field which can be supported by the conductor under consideration. This field was found in Sec. 2 for $\sigma = \text{const}$. This method is not valid in the nonlinear problem due to nonavailability of exact expressions for the solutions. Let problem (4.1) be solved for $H(z_0) = 1$, $q(z_0) = 0$, $H(\infty) = Q(\infty) = 0$. The z_0 value is determined by the supplementary condition

$$Q(z, z_0)|_{z=z_0} = Q^*.$$

A connection was indicated in Sec. 2 between the dimensional Q^*_{dim} and the dimensionless Q^*:

$$Q^*_{\text{dim}}/\mu H_0^2 = Q^* = Q(z_0, z_0) \equiv \varphi(z_0).$$

It is clear that the field is maximum when $z_0 \to \infty$. Consequently, the problem reduces to finding the limiting values of $\varphi(z_0)$ when $z_0 \to \infty$. The maximum field $H_{0\,\max}$ is then expressed by the equation

$$H_{0\max} = \sqrt{\frac{Q^*_{\rm dim}}{\mu \lim\limits_{z_0\to\infty} \varphi(z_0)}}.$$

We put in (4.1) $(z - z_0)/z_0 = \zeta$, $q = z_0\tilde{q}$, $E = z_0\tilde{E}$, $\varepsilon = z_0^{-2}$. Following the replacements indicated, we obtain the singular perturbation boundary value problem

$$\varepsilon\frac{dH}{d\zeta} = -\frac{\tilde{E}}{1+\beta Q}, \quad \varepsilon\frac{d\tilde{E}}{d\zeta} = -\frac{1+\zeta}{2}\frac{\tilde{E}}{1+\beta Q} - \varepsilon\frac{n_0\tilde{E}}{1+\zeta},$$
$$\varepsilon\frac{d\tilde{q}}{d\zeta} = -\frac{1+\zeta}{2\varkappa}\tilde{q} + \frac{\tilde{E}^2}{1+\beta Q}, \quad \varepsilon\frac{dQ}{d\zeta} = -\frac{q}{\varkappa}, \tag{4.4}$$
$$H(0) = 1,\ H(\infty) = Q(\infty) = 0,\ q(0) = 0.$$

We introduce the stretching variable $\eta = \zeta/\varepsilon$ [7] and consider (4.4) for $0 \leqslant \eta \leqslant \eta_0 < \infty$, $\varepsilon \to 0$. Retaining only the main terms, we find

$$\frac{dH}{d\eta} = -\frac{\tilde{E}}{1+\beta Q}, \quad \frac{d\tilde{E}}{d\eta} = -\frac{1}{2}\frac{\tilde{E}}{1+\beta Q},$$
$$\frac{d\tilde{q}}{d\eta} = -\frac{\tilde{q}}{2\varkappa} + \frac{\tilde{E}^2}{1+\beta Q}, \quad \varkappa\frac{dQ}{d\eta} = -\tilde{q}. \tag{4.5}$$

The system (4.5) has two first integrals:

$$\varkappa\frac{dQ}{d\eta} + \frac{1}{2}Q = \tilde{E}^2 + C_1, \quad \frac{H}{2} - \tilde{E} = C_2.$$

At $\eta \to \infty$ the solution of system (4.5) must be matched with the solution of system (4.4) [7], which is exponentially small for $\varepsilon \to 0$, $\zeta > 0$. This matching is possible only under the conditions $C_1 = C_2 = 0$. Putting then $\eta = 0$ and taking into account that $dQ/d\eta = 0$, for $\eta = 0$ we finally find

$$\lim_{z_0\to\infty} \varphi(z_0) = Q(0) = \frac{1}{2}.$$

The value of the maximum magnetic field which can be supported by the conductor has the same value as in the linear problem [see (2.8)]:

$$H_{0\max} = \sqrt{\frac{2Q^*_{\rm dim}}{\mu}}. \tag{4.6}$$

Equation (4.6) has a simple physical meaning: a conductor breakdown occurs if the magnetic energy density becomes equal to the coupling energy of the components of the conductor particles. Equation (4.6) for the linear problem (2.1) ($\sigma = \sigma_0$ = const) was obtained in many studies [1, 5, 8, 9].

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